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Equilibrium Points in Phase Spaces

Ananthkrishnan A
Manthan School,
Hyderabad

Ananya Sudakar
Vana Vani Mat. Hr. Sec. School,
Chennai

Mentor
Prof. Prasanta Kumar Tripathy

Equilibrium points:

In the sense of a phase space, an equilibrium point is a point with 0 velocity and constant displacement. If left undisturbed, a body stays in equilibrium for an infinite amount of time. The different types of equilibrium are classified by the motion of the body when a small force is applied on it. The 3 main classes are stable, unstable and neutral equilibrium. In stable equilibrium, a small force causes the body to move and return to its original position, as force is in the opposite direction of displacement. In unstable equilibrium, a small displacement leads the body to move further and further away from the equilibrium point. Another example is neutral equilibrium, where a displacement causes the body to move and form a new equilibrium point. A combination of these 3 can form many other types of equilibrium. A few include horse saddle equilibrium, where there is stable equilibrium in one direction, and unstable in all the other ones, and Mexican hat equilibrium, where there is a ring of neutral equilibrium, and stable equilibrium on either side. These are only possible in 3 dimensions, which we will not go into in the report.

What is a phase space?

A phase space is an arbitrary space in which all the possible states of a system can be represented using the knowledge of two dynamic variables. The two dynamic variables are position and velocity (or momentum). In one dimension, one axis will display position and another, velocity. For example, in one dimension, you would need one axis to display position and one for velocity, which is mainly what we dealt with. When moving to higher dimensions the phase space becomes harder to visualize. In three dimensions 3 axes will be required for position and 3 for velocity. This will mean that the phase space will have 6 axes.

Phase trajectories and phase portraits

The velocity and position of a body changes as time progresses. This graph can be plotted in the phase space which forms a relation between position and velocity, independent of time. The curve that you get by joining all the points is called the phase trajectory of the body and the set of all possible phase trajectories of a body is called the phase portrait.

Properties of phase trajectories

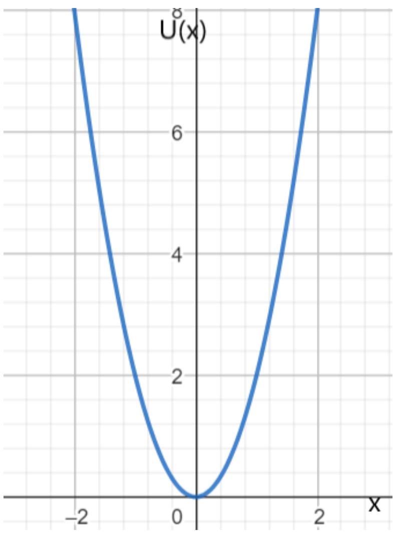
<u>Phase trajectories can't intersect themselves</u> - Phase trajectories are time independent. Due to this, at any particular initial state, the body must have only one unique trajectory that it follows. Assuming there is an intersection, there are 2 possible paths that the body could take from the point of intersection, which violates the fact that there can be only one unique path. Hence an intersection is not possible.

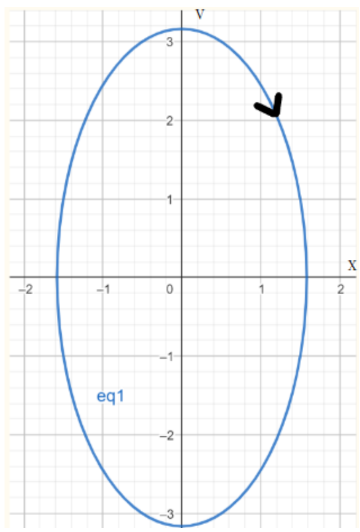
<u>Energy surface</u> - Assuming that total energy of the system is constant, the possible states of the system are limited in the sense that the sum of the kinetic and potential energy must be equal to the total energy. The possible states occupy a subspace of the phase space that is called the energy surface.

<u>Periodic motion</u> - Despite the fact that intersections are not possible in the graph, it can however form a closed loop. A closed loop implies that the body has periodic motion, as the same cycle is repeated, and the body reaches the initial state after a specific "time period".

Simple Harmonic Oscillator

Simple harmonic oscillator is an example of a stable equilibrium. When a body is displaced to a small extent and comes back to its original state it is in stable equilibrium. A simple pendulum is an example of stable equilibrium.The potential energy of the system in simple harmonic oscillator is $U(x) = \frac{kx^2}{2}$. The total energy is denoted by $\frac{mv^2}{2} + \frac{kx^2}{2} = E$, where $\frac{mv^2}{2}$ is the kinetic energy. On rearranging the equation we get $\frac{x^2}{2E/k} + \frac{v^2}{2E/m} = 1$ which is in the form of an equation for an ellipse.

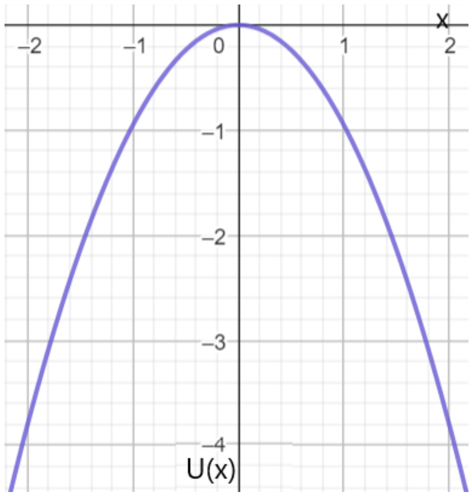


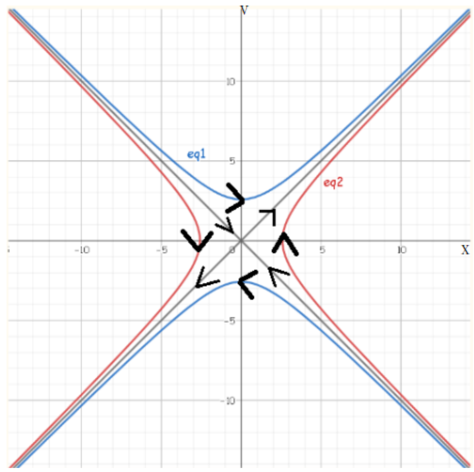


Inverted Harmonic Oscillator

This situation represents a hypothetical situation with potential energy $U(x) = \frac{-kx^2}{2}$. The total energy would be $\frac{mv^2}{2} - \frac{kx^2}{2} = E$, where $\frac{mv^2}{2}$ is the kinetic energy. The equation is rearranged to get $\frac{v^2}{2E/m} - \frac{x^2}{2E/k} = 1$, which is in the form of an equation for a hyperbola.

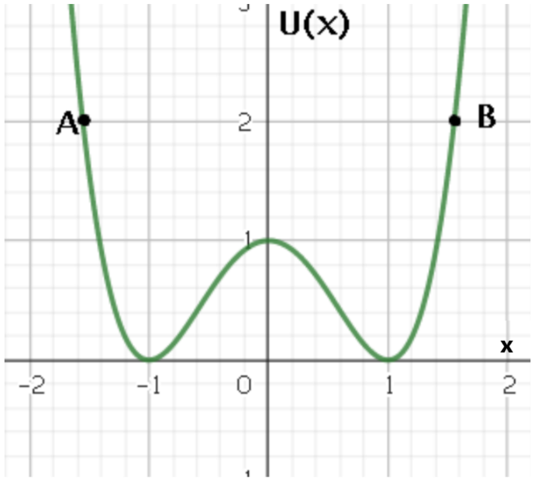
The orientation of the hyperbola depends on the value of E. If E is greater than 0, the axis of the hyperbola is x = 0 and vice versa. If, however, E = 0, an interesting situation appears. If a body is moving towards the peak of the potential energy curve, it is constantly decelerating, and the value of the deceleration increases as well. This means that the velocity tends to zero, but never reaches 0, taking an infinite amount of time to reach the origin. This explains the apparent intersection of the straight lines in the figure below.

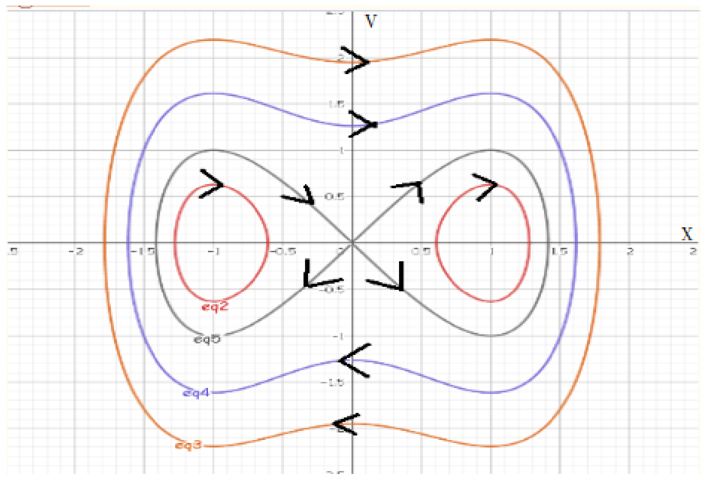




Miscellaneous Example 1:

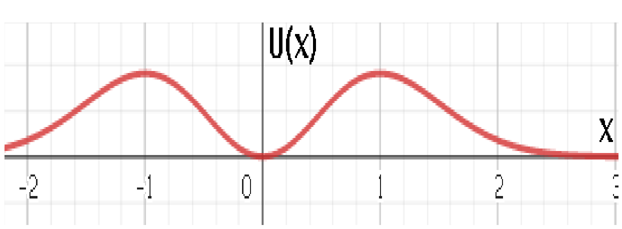
Let us take potential energy $U(x) = (x^2 - a^2)^2$. The potential energy curve has 2 local minima in between which there is a local maxima. Beyond the local minima, the curve goes to infinity. Using math we will not discuss in this report, we proved that at any local minima the curve behaves as a harmonic oscillator, and at a local maxima, the curve behaves as an inverted harmonic oscillator. Using this we can say that the phase trajectories also behave in similar fashion. If the total energy is not enough to cross the local maxima, the curve gets trapped in a harmonic oscillator fashion, around the local minima which you can see as ellipses. If the energy is enough to overcome the local maxima, it can move from, say point A to point B shown on the graph below. This corresponds to the outermost phase trajectories. At A and B the velocity would be 0, as the potential energy is then equal to total energy. There is a third case where the total energy is exactly equal to the potential energy at the local maxima. In this case, like the inverted harmonic oscillator, the graph approaches the origin, but takes infinite time to do so.

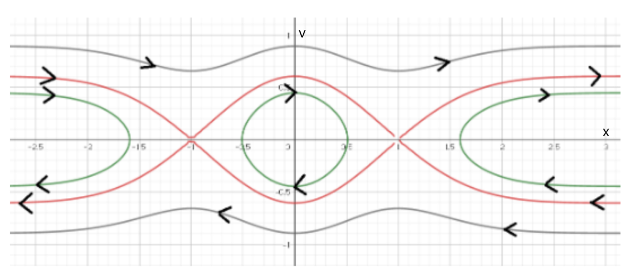




Miscellaneous Example 2:

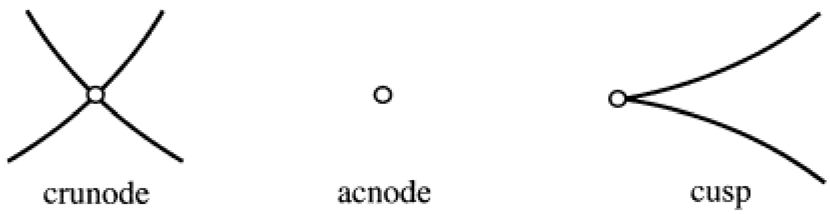
Now, let us take potential energy $U(x) = x^2 e^{-x^2}$. In this case, the potential energy curve has 2 local maxima, in between which there is a local minima. On either side of the local maxima, the curve tends to zero. There are a few cases to explore here. If the total energy is less than that of the local maxima, the curve can follow one of 3 cases. If it is beyond the maxima, it can rise up to a certain level along the curve, and then fall back down. This is represented by the 2 parabolas on either side. If it is in between the maxima, it takes an elliptical phase trajectory, shown in the middle. If energy is greater than that of the local maxima, it can pass over the maxima, down through the minima, and over the next maxima as well. After this, it moves for an infinite amount of time, with potential energy approaching zero (kinetic energy approaching total energy). These are shown as the topmost and bottommost graphs. If energy is equal to that of the maxima, it yet again approaches the maxima, but takes infinite time to do so.





Singularities

A singular point is a point on a curve where the tangent can not clearly be defined. At singular points the curve cannot be differentiated. One class of singular points is a double point. A double point is a point through which 2 branches of the curve pass. At double points, we can represent the slope ‘m’ in a quadratic $c_0+c_1m+c_2m^2$. If m has 2 distinct real solutions, the curve is smooth and non-singular. If, however, the roots are imaginary, it is called a crunode, where the curve intersects itself. If there is a single root, it is called a cusp, which is a sharp turning point. If there are imaginary roots, it is called an isolated singularity (acnode).



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